Isotropic cosmology in metric-affine gauge theory of gravity

A.V. Minkevich and A.S. Garkun†

Abstract

Geometrical structure of homogeneous isotropic models in the frame of the metric-affine gauge theory of gravity (MAGT) is analyzed. By using general form of gravitational Lagrangian including both a scalar curvature and various invariants quadratic in the curvature, torsion and nonmetricity tensors, gravitational equations of MAGT for homogeneous isotropic models are deduced. It is shown, that obtained gravitational equations lead to generalized cosmological Friedmann equation for the metrics by certain restrictions on indefinite parameters of gravitational Lagrangian. Isotropic models in the Weyl-cartan space-time are discussed.

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1 Introduction

The application of gauge approach to gravity leads to generalization of einsteinian general relativity theory (GR) [1]. There are different gauge theories of gravity (GT) in dependence on the choice of gravitational gauge group and gravitational Lagrangian: Poincare gauge theory (PGT), metric-affine gauge theory (MAGT), its simplest variant — gauge theory in the Weyl-Cartan space-time (WCGT) et al. Gauge theories of gravity were applied to resolve the problem of cosmological singularity of GR (see [2] and refs. given here). Regular isotropic cosmological models were built in the frame of PGT, MAGT (WCGT) by using sufficiently general gravitational Lagrangian L_G including both a scalar curvature and terms quadratic in the curvature and torsion tensors. Although the number and structure of gravitational equations of PGT, MAGT, WCGT are essentially different, as it was shown these GT lead to identical cosmological equations for the metrics in the case of homogeneous isotropic models [3, 4]. Their investigation allowed to make the following general conclusion. Satisfying the correspondence principle with GR in the case of gravitating systems with rather small energy densities, GT lead to essentially different physical consequences in comparison with GR in the case of gravitating systems at extreme conditions with extremely high energy densities

†Address: Department of Theoretical Physics, Belarussian State University, av. F.Skorina 4, 220050 Minsk, Republic of Belarus
E-mail: mink@phys.bsu.unibel.by

and pressures, the dynamics of which depends significantly on terms of L_G quadratic in the curvature tensor.

In the present paper further study of homogeneous isotropic models in the frame of MAGT (WCGT) is given by using more general gravitational Lagrangian including also terms quadratic in the nonmetricity tensor. In Sec. 2 geometrical structure of homogeneous isotropic space in MAGT is discussed. In Sec. 3 gravitational Lagrangian and gravitational equations of MAGT (WCGT) are given. In Sec. 4 gravitational equations of MAGT (WCGT) for homogeneous isotropic models are deduced. In Sec. 5 the generalized cosmological Friedmann equation is introduced and certain simplest isotropic models are discussed.

2 Geometrical structure of homogeneous isotropic models in MAGT

Geometrical structure of space-time in the frame of MAGT is defined by three tensors[1]: metrics $g_{\mu\nu}$, torsion $S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]}$ ($\Gamma^{\lambda}{}_{\mu\nu}$ is space-time connection), nonmetricity $Q_{\mu\nu\lambda} = \overset{\Gamma}{\nabla}_{\lambda} g_{\mu\nu}$ ($\overset{\Gamma}{\nabla}$ is the covariant derivative defined by $\Gamma^{\lambda}{}_{\mu\nu}$). The connection $\Gamma^{\lambda}{}_{\mu\nu}$ can be expressed by means of the Christoffel symbols $\left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\}$, torsion and nonmetricity as follows

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\} + S^{\lambda}{}_{\mu\nu} + S_{\mu\nu}{}^{\lambda} + S_{\nu\mu}{}^{\lambda} + \frac{1}{2}\left(Q_{\mu\nu}{}^{\lambda} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu}\right) \tag{1}$$

Let us consider the form of tensors $S^{\lambda}{}_{\mu\nu}$ and $Q_{\mu\nu\lambda}$ in the case of homogeneous isotropic models in the comoving reference frame by using spherical coordinates ($x^1 = r,\ x^2 = \vartheta,\ x^3 = \varphi$) and metrics in the form of the Robertson-Walker metrics

$$g_{\mu\nu} = diag\left(1,\ -\frac{R^2(t)}{1-kr^2},\ -R^2(t)\,r^2,\ -R^2(t)\,r^2\sin^2\vartheta\right), \tag{2}$$

where $R(t)$ is a scale factor, $k = -1, 0, +1$ in case of open, flat and close models respectively. The torsion tensor $S^{\lambda}{}_{\mu\nu}$ is determined by two functions of time $S(t)$ and $\tilde{S}(t)$, which define the following nonvanishing components [3]

$$\begin{aligned} &S^{1}{}_{10} = S^{2}{}_{20} = S^{3}{}_{30} = S(t)\,, \\ &S_{123} = S_{231} = S_{312} = \tilde{S}(t)\,\frac{R^3 r^2}{\sqrt{1-kr^2}}\sin\vartheta. \end{aligned} \tag{3}$$

By supposing that the theory is invariant under space inversions we have $\tilde{S}(t) = 0$ and the torsion tensor is defined by the only function $S(t)$. The nonmetricity tensor is defined by three

[1] μ, ν, ... are holonomic (world) indices; i, k ... are anholonomic (tetrad) indices. Numerical tetrad indices are denoted by means of a sign "ˆ" over them.

functions of time $Q_i(t)$ $(i=1,\ 2,\ 3)$ [5], so we have the following nonvanishing components:

$$\begin{gathered} Q_{000}=Q_2(t),\ Q_{110}=\frac{Q_1R^2}{1-kr^2},\ Q_{220}=Q_1R^2r^2, \\ Q_{330}=Q_1R^2r^2\sin^2\vartheta,\ Q_{011}=\frac{Q_3R^2}{1-kr^2}, \\ Q_{022}=Q_3R^2r^2,\ Q_{033}=Q_3R^2r^2\sin^2\vartheta. \end{gathered} \tag{4}$$

By choosing the tetrad corresponding to the metrics (2) in the diagonal form

$$\begin{gathered} h^i{}_\mu=diag\left(1,\ \tfrac{R(t)}{\sqrt{1-kr^2}},\ R\left(t\right)r,\ R\left(t\right)r\sin\vartheta\right), \\ \left(g_{\mu\nu}=\eta_{ik}h^i{}_\mu h^k{}_\nu,\ \eta_{ik}=diag\left(1,\ -1,\ -1,\ -1\right)\right), \end{gathered} \tag{5}$$

we can write nonvanishing anholonomic components of torsion and nonmetricity as

$$\begin{aligned} &S_{\hat{1}\hat{0}\hat{1}}=S_{\hat{2}\hat{0}\hat{2}}=S_{\hat{3}\hat{0}\hat{3}}=S\left(t\right), \\ &Q_{\hat{1}\hat{1}\hat{0}}=Q_{\hat{2}\hat{2}\hat{0}}=Q_{\hat{3}\hat{3}\hat{0}}=Q_1\left(t\right), \\ &Q_{\hat{0}\hat{0}\hat{0}}=Q_2\left(t\right), \\ &Q_{\hat{0}\hat{1}\hat{1}}=Q_{\hat{0}\hat{2}\hat{2}}=Q_{\hat{0}\hat{3}\hat{3}}=Q_3\left(t\right). \end{aligned} \tag{6}$$

Thus the geometrical structure of homogeneous isotropic models studied below is determined by five functions of time: $R(t)$, $S(t)$, $Q_i(t)$ $(i=1,2,3)$.

Using (1) – (6) one finds nonvanishing components of gauge potentials $A^{ik}{}_\mu$ as follows

$$A^{ik}{}_\mu=h^{k\nu}\left(\partial_\mu h^i{}_\nu-h^i{}_\lambda\Gamma^\lambda{}_{\nu\mu}\right). \tag{7}$$

So, for $A^{ik}{}_l=h_l{}^\mu A^{ik}{}_\mu$ we have

$$\begin{aligned} &A^{\hat{0}\hat{0}}{}_{\hat{0}}=\tfrac{1}{2}Q_2,\ A^{\hat{1}\hat{1}}{}_{\hat{0}}=A^{\hat{2}\hat{2}}{}_{\hat{0}}=A^{\hat{3}\hat{3}}{}_{\hat{0}}=\tfrac{1}{2}Q_1, \\ &A^{[\hat{0}\hat{1}]}{}_{\hat{1}}=A^{[\hat{0}\hat{2}]}{}_{\hat{2}}=A^{[\hat{0}\hat{3}]}{}_{\hat{3}}=\frac{\dot{R}-2R\left(S-\frac{1}{4}Q_1+\frac{1}{4}Q_3\right)}{R}, \\ &A^{[\hat{1}\hat{2}]}{}_{\hat{2}}=A^{[\hat{1}\hat{3}]}{}_{\hat{3}}=-\frac{\sqrt{1-kr^2}}{Rr},\ A^{[\hat{3}\hat{2}]}{}_{\hat{3}}=\frac{\cot\vartheta}{Rr}, \\ &A^{(\hat{0}\hat{1})}{}_{\hat{1}}=A^{(\hat{0}\hat{2})}{}_{\hat{2}}=A^{(\hat{0}\hat{3})}{}_{\hat{3}}=-\tfrac{1}{2}Q_3, \end{aligned} \tag{8}$$

where a dot means differentiation with respect to time. Taking into account (8) it is not difficult to find the curvature tensor $F^{ik}{}_{\mu\nu}$ defined by

$$F^{ik}{}_{\mu\nu}=2\partial_{[\mu}A^{ik}{}_{\nu]}+2A^i{}_{l[\nu}A^{lk}{}_{\mu]}. \tag{9}$$

Nonvanishing components of this tensor are given by three curvature functions A, B, C:

$$\begin{aligned} &F^{01}{}_{01}=F^{02}{}_{02}=F^{03}{}_{03}=A-C, \\ &F^{10}{}_{10}=F^{20}{}_{20}=F^{30}{}_{30}=A+C, \\ &F^{12}{}_{12}=F^{13}{}_{13}=F^{23}{}_{23}=B, \end{aligned} \tag{10}$$

where

$$
\begin{aligned}
A &= \frac{\left[\dot{R} - 2RS_q\right]^{\cdot}}{R} + \frac{1}{4}Q_3\,(Q_1 + Q_2),\\
B &= \frac{k + \left[\dot{R} - 2RS_q\right]^2}{R^2} - \frac{1}{4}Q_3^2,\\
C &= \frac{1}{2}\frac{\dot{R} - 2RS_q}{R}\,(Q_1 + Q_2) + \frac{1}{2}\frac{(RQ_3)^{\cdot}}{R},\\
S_q &= S - \tfrac{1}{4}Q_1 + \tfrac{1}{4}Q_3,
\end{aligned}
\tag{11}
$$

and $F^{[01]}{}_{01} = A$, $F^{(01)}{}_{01} = -C$, $F^{(12)}{}_{12} = 0$. In accordance with (10) one obtains nonvanishing components of the tensors $F^{\mu}{}_{\nu} = F^{\lambda\mu}{}_{\lambda\nu}$ and $\tilde{F}^{\mu}{}_{\nu} = F^{\mu\lambda}{}_{\nu\lambda}$:

$$
\begin{aligned}
&F^0{}_0 = 3\,(A + C)\,,\;\; F^1{}_1 = F^2{}_2 = F^3{}_3 = A + 2B - C,\\
&\tilde{F}^0{}_0 = 3\,(A - C)\,,\;\; \tilde{F}^1{}_1 = \tilde{F}^2{}_2 = \tilde{F}^3{}_3 = A + 2B + C.
\end{aligned}
\tag{12}
$$

All components of the tensor $V_{\mu\nu} = F^{\lambda}{}_{\lambda\mu\nu}$ are equal to zero. The scalar curvature $F = F^{\mu}{}_{\mu} = \tilde{F}^{\mu}{}_{\mu} = 6\,(A + B)$. The Bianchy identities for homogeneous isotropic models are reduced to the following relation

$$
\dot{B} + 2H\,(B - A) + 4AS_q + CQ_3 = 0, \qquad \left(H = \frac{\dot{R}}{R}\right). \tag{13}
$$

3 Gravitational Lagrangian and gravitational field equations

In the frame of MAGT the role of gauge potentials play the tetrad $h^i{}_{\mu}$, anholonomic connection $A^{ik}{}_{\mu}$ and anholonomic metrics [1]. The corresponding field strengths are the tensors $S^i{}_{\mu\nu}$, $F^{ik}{}_{\mu\nu}$ and $Q_{ik\mu}$. For mathematical simplification of further analysis we shall suppose the tetrad to be orthonormalized. Thus the anholonomic metrics is fixed in the form of Minkowski metrics η_{ij}. Then the torsion and nonmetricity tensors can be represented as functions of gauge potentials

$$
\begin{aligned}
S^i{}_{\mu\nu} &= \partial_{[\nu}h^i{}_{\mu]} - h_{k[\mu}A^{ik}{}_{\nu]},\\
Q^{ik}{}_{\mu} &= 2A^{(ik)}{}_{\mu}.
\end{aligned}
\tag{14}
$$

The curvature tensor is determined by eqs. (9).

The gravitational Lagrangian in MAGT is a function of $h^i{}_{\mu}$, $S^i{}_{\mu\nu}$, $F^{ik}{}_{\mu\nu}$ and $Q^{ik}{}_{\mu}$

$$
L_G = h\mathcal{L}_G\left(h^i{}_{\mu}, S^i{}_{\mu\nu}, F^{ik}{}_{\mu\nu}, Q^{ik}{}_{\mu}\right),\;\; h = \det\left(h^i{}_{\mu}\right), \tag{15}
$$

where $\mathcal{L}_G$ is invariant. The gravitational equations can be derived by variation of the total action integral $I = \int d^4x\,(L_m + L_G)$ (L_m is Lagrangian of matter) with respect to $h^i{}_{\mu}$ and $A^{ik}{}_{\mu}$. In connection with this the obtained system includes two groups of differential equations — 16 h-eqs. and 64 A-eqs. :

$$
H_i{}^{\mu} - \nabla_{\nu}\sigma_i{}^{\mu\nu} = t_i{}^{\mu}, \tag{16}
$$

$$2\nabla_\nu \varphi_{ik}{}^{\nu\mu} + \sigma_{ik}{}^{\mu} - 2\varphi_{il}{}^{\nu\mu} Q^l{}_{k\nu} - 2\Omega_{ik}{}^{\mu} = -J_{ik}{}^{\mu}, \tag{17}$$

where the following notations are introduced

$$H_i{}^\mu = \frac{1}{h}\frac{\partial L_G}{\partial h^i{}_\mu},\ \sigma_i{}^{\mu\nu} = \frac{\partial \mathcal{L}_G}{\partial S^i{}_{\mu\nu}},\ \varphi_{ik}{}^{\mu\nu} = \frac{\partial \mathcal{L}_G}{\partial F^{ik}{}_{\mu\nu}},$$
$$\Omega_{ik}{}^\mu = \frac{\partial \mathcal{L}_G}{\partial Q^{ik}{}_\mu},\ t_i{}^\mu = -\frac{1}{h}\frac{\delta L_m}{\delta h^i{}_\mu},\ J_{ik}{}^\mu = -\frac{1}{h}\frac{\delta L_m}{\delta A^{ik}{}_\mu},$$

and ∇ denotes here covariant derivative defined by means of $(-A^{ik}{}_\mu)$ and $\left\{ \begin{array}{c} \lambda \\ \mu\,\nu \end{array} \right\}$in the case of tetrad and world indices respectively (for example $\nabla_\nu h^i{}_\mu = \partial_\nu h^i{}_\mu A^i{}_{l\nu} h^l{}_\mu - \left\{ \begin{array}{c} \lambda \\ \mu\,\nu \end{array} \right\} h^i{}_\lambda$).

The system of 64 A-equations (17) can be divided into two groups of equations — 24 $A^{[ik]}{}_\mu$-eqs. and 40 $A^{(ik)}{}_\mu$-eqs. corresponding to variation with respect to antisymmetric and symmetric parts of gravitational potentials $A^{ik}{}_\mu$. According to (1) and (7) we have

$$A^{[ik]}{}_\mu = \overset{(R.-C.)}{A}{}^{[ik]}{}_\mu + Q_\mu{}^{[ik]}, \tag{18}$$

where $\overset{(R.-C.)}{A}{}^{[ik]}{}_\mu$ as function of tetrad and torsion has the same form as in the Riemann-Cartan space-time, and the symmetric part is defined by (14).

In the frame of WCGT the nonmetricity tensor is given by

$$Q^{ik}{}_\mu = \eta^{ik} Q_\mu, \tag{19}$$

where Q_μ is the Weyl's vector. By using the gravitational Lagrangian of WCGT in the form (15) we obtain the gravitational equations of WCGT by variation of the total action with respect to $h^i{}_\mu$, $A^{[ik]}{}_\mu$ and Q_μ. As result we have h-eqs. in the form (16), and $A^{[ik]}{}_\mu$-eqs. and Q_μ-eqs. in the following form

$$2\nabla_\nu \varphi_{[ik]}{}^{\nu\mu} + \sigma_{[ik]}{}^\mu - 2\varphi_{[ik]}{}^{\nu\mu} Q_\nu = -J_{ik}{}^\mu, \tag{20}$$

$$2\nabla_\nu \varphi_i{}^{i\nu\mu} + \sigma_i{}^{i\mu} - 2\Omega_i{}^{i\mu} = -J_i{}^{i\mu}. \tag{21}$$

The system of $A^{[ik]}{}_\mu$-eqs. (20) is antisymmetric part of A-eqs. (17) and Q_μ-eqs. (21) can be obtained by contraction of (17) with the tensor η^{ik}.

The explicit form of eqs (16)-(17) and also (20) – (21) depends on the choice of the gravitational Lagrangian. Because the explicit form of L_G in MAGT is unknown, we shall use sufficiently general expression including both a scalar curvature and different invariants quadratic in the tensors $F^{ik}{}_{\mu\nu}$, $S^i{}_{\mu\nu}$ and $Q^{ik}{}_\mu$ built by taking into account their symmetry properties:

$$\mathcal{L}_G = \mathcal{L}_F + \mathcal{L}_S + \mathcal{L}_Q + \mathcal{L}_{QS}, \tag{22}$$

where

$$
\begin{aligned}
\mathcal{L}_F &= f_0 F + F^{\alpha\beta\mu\nu}\left(f_1 F_{\alpha\beta\mu\nu} + f_2 F_{\beta\alpha\mu\nu} + f_3 F_{\alpha\mu\beta\nu} + f_4 F_{\beta\mu\alpha\nu} +\right. \\
&\quad \left. + f_5 F_{\mu\nu\alpha\beta}\right) + f_6 F^2 + F^{\mu\nu}\left(f_7 F_{\mu\nu} + f_8 F_{\nu\mu} + f_9 \tilde{F}_{\mu\nu} + f_{10}\tilde{F}_{\nu\mu}\right) + \\
&\quad + \tilde{F}^{\mu\nu}\left(f_{11}\tilde{F}_{\mu\nu} + f_{12}\tilde{F}_{\nu\mu}\right) + V^{\mu\nu}\left(f_{13}F_{\mu\nu} + f_{14}\tilde{F}_{\mu\nu} + f_{15}V_{\mu\nu}\right), \\
\mathcal{L}_S &= S^{\alpha\mu\nu}\left(a_1 S_{\alpha\mu\nu} + a_2 S_{\nu\mu\alpha}\right) + a_3 S^{\alpha}{}_{\mu\alpha} S_{\beta}{}^{\mu\beta}, \\
\mathcal{L}_Q &= k_1 Q_{\mu\nu\lambda}Q^{\mu\nu\lambda} + k_2 Q_{\mu\nu\lambda}Q^{\mu\lambda\nu} + k_3 Q^{\mu}{}_{\mu\lambda}Q^{\nu}{}_{\nu}{}^{\lambda} + \\
&\quad + k_4 Q^{\mu}{}_{\lambda\mu}Q^{\nu\lambda}{}_{\nu} + k_5 Q^{\mu}{}_{\mu\lambda}Q^{\nu\lambda}{}_{\nu}, \\
\mathcal{L}_{QS} &= m_1 Q_{\mu\nu\lambda}S^{\mu\nu\lambda} + m_2 Q^{\alpha}{}_{\alpha\lambda}S_{\beta}{}^{\beta\lambda} + m_3 Q^{\alpha}{}_{\lambda\alpha}S_{\beta}{}^{\beta\lambda}.
\end{aligned}
\tag{23}
$$

The correspondence principle with GR leads to $f_0 = (16\pi G)^{-1}$ (G is Newton's gravitational constant). The gravitational Lagrangian contains a large number of indefinite parameters f_i ($i = 1, \ldots, 15$), a_l ($l = 1, 2, 3$), k_p ($p = 1, \ldots, 5$) m_s ($s = 1, 2, 3$). Some restrictions on these parameters will be found below as result of analysis of homogeneous isotropic models.

The tensors $H^i{}_\mu$, $\varphi_{ik}{}^{\mu\nu}$, $\sigma_i{}^{\mu\nu}$ and $\Omega_{ik}{}^{\mu}$ corresponding to Lagrangian (22)–(23) have the following form:

$$
\begin{aligned}
&H_i{}^{\mu} = -\left(f_0 + 2f_6 F\right)\left(F^{\mu}{}_i + \tilde{F}^{\mu}{}_i\right) + 4f_1 F_{klmi}F^{kl\mu m} + 4f_2 F^{kl\mu m}F_{lkmi} + \\
&+ 4f_3 F_{klmi}F^{k[m\mu]l} + 2f_4 F_{klmi}\left(F^{l[m\mu]k} + F^{[\mu|kl|m]}\right) + 4f_5 F_{klmi}F^{[\mu m]kl} - \\
&- 2f_7\left(F^{k\mu}F_{ki} - F^{kl}F^{\mu}{}_{kli}\right) - 2f_8\left(F^{\mu k}F_{ki} - F^{kl}F^{\mu}{}_{lki}\right) - f_9\left(\tilde{F}^{k\mu}F_{ki} + \tilde{F}_{ki}F^{k\mu} + \right. \\
&\left. + F^{k\mu l}{}_i F_{kl} - F^{\mu}{}_{kli}\tilde{F}^{kl}\right) - f_{10}\left(\tilde{F}^{\mu k}F_{ki} + \tilde{F}_{ki}F^{\mu k} + F^{k\mu l}{}_i F_{lk} - F^{\mu}{}_{lki}\tilde{F}^{kl}\right) - \\
&- 2f_{11}\left(\tilde{F}_{ki}\tilde{F}^{k\mu} + F^{k\mu l}{}_i\tilde{F}_{kl}\right) - 2f_{12}\left(\tilde{F}_{ki}\tilde{F}^{\mu k} + F^{k\mu l}{}_i\tilde{F}_{lk}\right) + f_{13}\left(2V_{ik}F^{[k\mu]} - F_{ki}V^{k\mu} + \right. \\
&\left. + V^{kl}F^{\mu}{}_{kli}\right) + f_{14}\left(2V_{ik}\tilde{F}^{[k\mu]} - \tilde{F}_{ki}V^{k\mu} - V_{kl}F^{k\mu l}{}_i\right) - 4f_{15}V_{ik}V^{\mu k} - 4S^{k}{}_{i\nu}\left(a_1 S_k{}^{\mu\nu} - \right. \\
&\left. - a_2 S^{\mu\nu}{}_k - a_3 S^{\alpha}{}_{\alpha}{}^{[\mu}h_k{}^{\nu]}\right) - 2k_1 Q^{kl\mu}Q_{kli} - 2k_2 Q^{k\mu l}Q_{kli} - 2k_3 Q^{k}{}_{k}{}^{\mu}Q^{l}{}_{li} - \\
&- 2k_4 Q^{k\mu}{}_i Q^{lk}{}_l - 2k_5 Q^{k}{}_{k(l}Q^{l\mu}{}_{i)} - m_1\left(Q_{kli}S^{kl\mu} - 2Q^{k[\mu l]}S_{kli}\right) - m_2\left(Q^{k}{}_{k}{}^{l}S^{\mu}{}_{il} + \right. \\
&\left. + Q^{k}{}_{ki}S^{l}{}_{l}{}^{\mu} + Q^{k}{}_{k}{}^{\mu}S^{l}{}_{li}\right) - m_3\left(2Q^{\mu k}{}_{(i}S^{m}{}_{|m|k)} + Q^{kl}{}_k S^{\mu}{}_{il}\right) - h^i{}_\mu \mathcal{L}_G,
\end{aligned}
\tag{24}
$$

$$
\sigma_i{}^{\mu\nu} = 2\left(a_1 S_i{}^{[\mu\nu]} - a_2 S^{\mu\nu}{}_i - a_3 S^{\alpha}{}_{\alpha}{}^{[\mu}h_i{}^{\nu]}\right) + m_1 Q_i{}^{[\mu\nu]} + \left(m_2 Q^{\lambda}{}_{\lambda}{}^{[\nu} + m_3 Q^{\lambda[\nu}{}_{\lambda}\right)h_i{}^{\mu]} \tag{25}
$$

$$
\begin{aligned}
&\varphi_{ik}{}^{\mu\nu} = \left(f_0 + 2f_6 F\right)h_i{}^{[\mu}h_k{}^{\nu]} + 2f_1 F_{ik}{}^{\mu\nu} + 2f_2 F_{ki}{}^{\mu\nu} + 2f_3 F_i{}^{[\mu}{}_k{}^{\nu]} + f_4\left(F_k{}^{[\mu}{}_i{}^{\nu]} + \right. \\
&\left. + F^{[\mu}{}_{ik}{}^{\nu]}\right) + 2f_5 F^{\mu\nu}{}_{ik} + 2f_7 F_k{}^{[\nu}h_i{}^{\mu]} + 2f_8 F^{[\nu}{}_k h_i{}^{\mu]} + f_9\left(\tilde{F}_k{}^{[\nu}h_i{}^{\mu]} + F_i{}^{[\mu}h_k{}^{\nu]}\right) + \\
&+ f_{10}\left(\tilde{F}^{[\nu}{}_k h_i{}^{\mu]} + F^{[\mu}{}_i h_k{}^{\nu]}\right) + 2f_{11}\tilde{F}_i{}^{[\mu}h_k{}^{\nu]} + 2f_{12}\tilde{F}^{[\mu}{}_i h_k{}^{\nu]} + f_{13}\left(F^{[\mu\nu]}\eta_{ik} + h_i{}^{[\mu}V_k{}^{\nu]}\right) + \\
&+ f_{14}\left(\tilde{F}^{[\mu\nu]}\eta_{ik} + V_i{}^{[\mu}h_k{}^{\nu]}\right) + 2f_{15}V^{\mu\nu}\eta_{ik},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\Omega_{ik}{}^{\mu} &= 2k_1 Q_{ik}{}^{\mu} + 2k_2 Q_{(i}{}^{\mu}{}_{k)} + 2k_3\eta_{ik}Q^{\nu}{}_{\nu}{}^{\mu} + 2k_4 h_{(i}{}^{\mu}Q^{\nu}{}_{k)\nu} + k_5\left(\eta_{ik}Q^{\nu\mu}{}_{\nu} + \right. \\
&\quad \left. + Q^{\nu}{}_{\nu(k}h_{i)}{}^{\mu}\right) + m_1 S_{(ik)}{}^{\mu} + m_2\eta_{ik}S^{\nu}{}_{\nu}{}^{\mu} + m_3 S^{\nu}{}_{\nu(k}h_{i)}{}^{\mu}.
\end{aligned}
\tag{27}
$$

4 Gravitational equations for homogeneous isotropic models in MAGT

Using the gravitational Lagrangian (22) and also (1) – (12) we derive the gravitational equations for homogeneous isotropic models. In accordance with the structure of the second and

third rank tensors in homogeneous isotropic space we obtain two h-equations ($h^{\hat{0}}{}_{0}$- $h^{\hat{1}}{}_{1}$-eqs.) and four A-equations ($A^{\hat{0}\hat{0}}{}_{0}$- ,$A^{\hat{1}\hat{1}}{}_{0}$- , $A^{(\hat{0}\hat{1})}{}_{1}$- , $A^{[\hat{0}\hat{1}]}{}_{1}$-eqs.). In order to simplify calculations REDUCE computer algebra system was used. The obtained equations have the following form:

$$\begin{aligned} t_{\hat{0}}{}^{\hat{0}} &= 6f_0 B - 12f\left(A^2-B^2\right) - 12C\left(f^{\mathrm{I}}C + f^{\mathrm{II}}A\right) + 3aS^2 - 3\left(k_1+3k_3\right)Q_1^2 + \\ &+ 3\left(2k_3+k_5\right)Q_1Q_2 - \left(k_1+k_2+k_3+k_4+k_5\right)Q_2^2 + 3\left(2k_1+k_2+3k_4\right)Q_3^2 + \\ &+ \tfrac{3}{2}\left[\left(m_1+3m_2\right)Q_1 - \left(m_2+m_3\right)Q_2 - \left(m_1-3m_3\right)Q_3 - 2aS\right]\left(H+\tfrac{1}{2}Q_1\right) + \\ &+ 3\left(m_1-3m_3\right)SQ_3, \end{aligned} \tag{28}$$

$$\begin{aligned} t_{\hat{1}}{}^{\hat{1}} &= 2f_0\left(2A+B\right) + 4f\left(A^2-B^2\right) + 4C\left(f^{\mathrm{I}}C + f^{\mathrm{II}}A\right) - 3aS^2 + 3\left(k_1+3k_3\right)Q_1^2 - \\ &- 3\left(2k_3+k_5\right)Q_1Q_2 + 2\left(2k_2+3k_5\right)Q_1Q_3 + \left(k_1+k_2+k_3+k_4+k_5\right)Q_2^2 - \\ &- 2\left(2k_4+k_5\right)Q_2Q_3 + \left(2k_1+k_2+3k_4\right)Q_3^2 + \left[\left(m_1+3m_2\right)Q_1 - \left(m_2+m_3\right)Q_2 - \right. \\ &\left. - \left(m_1-3m_3\right)Q_3 - 2aS\right]\left(H-2S-\tfrac{1}{4}Q_1\right) + \tfrac{1}{2}\left[\left(m_1+3m_2\right)\dot{Q}_1 - \tfrac{1}{2}\left(m_2+m_3\right)\dot{Q}_2 - \right. \\ &\left. - \tfrac{1}{2}\left(m_1-3m_3\right)\dot{Q}_3 - 2a\dot{S}\right], \end{aligned} \tag{29}$$

$$\begin{aligned} J_{\hat{0}\hat{0}}{}^{\hat{0}} &= -6\left(\tfrac{f_0}{2} + 2fA - f^{\mathrm{III}}B + f^{\mathrm{II}}C\right)Q_3 - 12\left(H-2S_q\right)\left(f^{\mathrm{II}}\left(A-B\right) + 2f^{\mathrm{I}}C\right) + \\ &+ 6\left(2k_3+k_5\right)Q_1 - 4\left(k_1+k_2+k_3+k_4+k_5\right)Q_2 + 6\left(2k_4+k_5\right)Q_3 - 6\left(m_2+m_3\right)S, \end{aligned} \tag{30}$$

$$\begin{aligned} J_{(\hat{0}\hat{1})}{}^{\hat{1}} &= -\left(\tfrac{f_0}{2} + 2fA - f^{\mathrm{III}}B + f^{\mathrm{II}}C\right)\left(Q_1+Q_2\right) + 2\left(\tfrac{f_0}{2} + 2fB - f^{\mathrm{III}}A - f^{\mathrm{II}}C\right)Q_3 + \\ &+ 4H\left[f^{\mathrm{II}}\left(A-B\right) + 2f^{\mathrm{I}}C\right] + 2\left[f^{\mathrm{II}}\left(\dot{A}-\dot{B}\right) + 2f^{\mathrm{I}}\dot{C}\right] - \left(2k_2+3k_5\right)Q_1 + \\ &+ \left(2k_4+k_5\right)Q_2 - 2\left(2k_1+k_2+3k_4\right)Q_3 - \tfrac{1}{4}\left[\left(m_1+3m_2\right)Q_1 - \left(m_2+m_3\right)Q_2 - \right. \\ &\left. - \left(m_1-3m_3\right)Q_3 - 2aS\right] - \left(m_1-3m_3\right)S, \end{aligned} \tag{31}$$

$$\begin{aligned} J_{\hat{1}\hat{1}}{}^{\hat{0}} &= 2\left(\tfrac{f_0}{2} + 2fA - f^{\mathrm{III}}B + f^{\mathrm{II}}C\right)Q_3 + 4\left(H-2S_q\right)\left(f^{\mathrm{II}}\left(A-B\right) + 2f^{\mathrm{I}}C\right) - \\ &+ 4\left(k_1+3k_3\right)Q_1 - 2\left(2k_3+k_5\right)Q_2 + 2\left(2k_2+3k_5\right)Q_3 - \tfrac{1}{2}\left[\left(m_1+3m_2\right)Q_1 - \right. \\ &\left. - \left(m_2+m_3\right)Q_2 - \left(m_1-3m_3\right)Q_3 - 2aS\right] - 2\left(m_1+3m_2\right)S, \end{aligned} \tag{32}$$

$$\begin{aligned} J_{[\hat{0}\hat{1}]}{}^{\hat{1}} &= 16f\left(\dot{A}+\dot{B}\right) + 16\left[f_0 + 4f\left(A+B\right)\right]S_q + 8\left(f^{\mathrm{III}}+2f\right)CQ_3 - \\ &- \left[\left(m_1+3m_2\right)Q_1 - \left(m_2+m_3\right)Q_2 - \left(m_1-3m_3\right)Q_3 - 2aS\right] + \\ &+ 8f^{\mathrm{II}}\dot{C} - 4\left[2f^{\mathrm{I}}C + f^{\mathrm{II}}\left(A-B\right)\right]\left(Q_1+Q_2\right) + 32f^{\mathrm{II}}\left(H-S_q\right)C, \end{aligned} \tag{33}$$

where the following notations are introduced:

$$\begin{aligned} a &= 2a_1 + a_2 + 3a_3, \\ f &= f_1 - f_2 + \tfrac{f_3-f_4}{2} + f_5 + 3f_6 + \sum_{i=7}^{12} f_i, \\ f^{\mathrm{I}} &= f_1 + f_2 + \tfrac{f_3+f_4}{2} + f_7 + f_8 - f_9 - f_{10} + f_{11} + f_{12}, \\ f^{\mathrm{II}} &= f_7 + f_8 - f_{11} - f_{12}, \\ f^{\mathrm{III}} &= 2f^{\mathrm{I}} - f^{\mathrm{II}} - 2f - 4f_2 - 2f_4 + 2f_5 + 3f_9 + 3f_{10} - 2f_{11} - 2f_{12} = \\ &= -6f_6 - f_7 - f_8 - f_9 - f_{10} - f_{11} - f_{12}. \end{aligned} \tag{34}$$

The system of gravitational equations (28) – (33) includes 12 coefficients f_i ($i = 1, \ldots, 12$) in the form of four combinations f, f^{I}, f^{II}, f^{III}. Three coefficients a_k ($k = 1, 2, 3$) appear in equations by means of a. Coefficients k_i and m_s appear in various combinations.

Further analysis of obtained system of equations is connected with searching of such restrictions on indefinite parameters, which essentially simplify investigated system and leads to cosmological equations, which are reasonable generalization of the cosmological Friedmann equations of GR.

As source of gravitational field we will consider fluid (field) with the following nonvanishing components of the energy-momentum tensor $t_{\hat{0}}{}^{\hat{0}} = \rho$, $t_{\hat{1}}{}^{\hat{1}} = t_{\hat{2}}{}^{\hat{2}} = t_{\hat{3}}{}^{\hat{3}} = -p$ (ρ is the energy (mass) density, p is the pressure) and with vanishing tensor $J_{ik}{}^{\mu} = 0$.

5 Generalized cosmological Friedmann equation and homogeneous isotropic models in the Weyl-Cartan space-time

At first, let us put $k_i = 0$ and $m_k = 0$. It means that terms of L_G including nonmetricity are omitted. Then the gravitational equations (28)–(33) can be transformed to the form

$$6f_0 B - 12f\left(A^2 - B^2\right) - 12C\left(f^{\mathrm{I}}C + f^{\mathrm{II}}A\right) + 3aS^2 - \tfrac{3}{2}a\left(Q_1 + 2H\right)S = \rho, \tag{35}$$

$$\begin{aligned} &2f_0\left(2A + B\right) + 4f\left(A^2 - B^2\right) + 4C\left(f^{\mathrm{I}}C + f^{\mathrm{II}}A\right) + aS^2 - a\dot{S} + \\ &+ \tfrac{1}{2}a\left(Q_1 - 4H\right)S = -p, \end{aligned} \tag{36}$$

$$\left[f_0 + 2f^{\mathrm{II}}C - 2f^{\mathrm{III}}B + 4fA\right]Q_3 + 4\left(H - 2S_q\right)\left[2f^{\mathrm{I}}C + f^{\mathrm{II}}\left(A - B\right)\right] = 0, \tag{37}$$

$$\begin{aligned} &\left[\tfrac{f_0}{2} + 2fA - f^{\mathrm{III}}B + f^{\mathrm{II}}C\right]\left(Q_1 + Q_2\right) - 2\left(\tfrac{f_0}{2} + 2fB - f^{\mathrm{III}}A - f^{\mathrm{II}}C\right)Q_3 - \\ &- 4\left(f^{\mathrm{II}}\left(A - B\right) + 2f^{\mathrm{I}}C\right)H - 2\left(f^{\mathrm{II}}\left(\dot{A} - \dot{B}\right) + 2f^{\mathrm{I}}\dot{C}\right) - \tfrac{1}{2}aS = 0, \end{aligned} \tag{38}$$

$$aS = 0 \tag{39}$$

$$\begin{aligned} &4f\left(\dot{A} + \dot{B}\right) + 4\left[f_0 + 4f\left(A + B\right)\right]S_q + 2\left(2f + f^{\mathrm{III}}\right)CQ_3 + \tfrac{1}{2}aS - \\ &- \left[f^{\mathrm{II}}\left(A - B\right) + 2f^{\mathrm{I}}C\right]\left(Q_1 + Q_2\right) + 2f^{\mathrm{II}}\left[\dot{C} + 4\left(H - S_q\right)C\right] = 0 \end{aligned} \tag{40}$$

If one imposes on the parameters two following conditions

$$f^{\mathrm{I}} = 0, \qquad f^{\mathrm{II}} = 0, \tag{41}$$

then equations (37) – (39) give

$$Q_1 + Q_2 = 0, \qquad Q_3 = 0. \tag{42}$$

By means of (42) and (41) equations (35), (36) and (40) take the form

$$\left.\begin{aligned} &6f_0 B - 12f\left(A^2 - B^2\right) = \rho \\ &2f_0\left(2A + B\right) + 4f\left(A^2 - B^2\right) = -p \\ &4f\left(\dot{A} + \dot{B}\right) + 4\left[f_0 + 4f\left(A + B\right)\right]S_q = 0 \end{aligned}\right\} \tag{43}$$

Equations (43) lead to the following solution

$$\begin{cases} S-\dfrac{1}{4}Q_1=-\dfrac{1}{4}\dfrac{d}{dt}\ln\left|1-\beta\left(\rho-3p\right)\right| \\ A=-\dfrac{1}{12f_0}\dfrac{\rho+3p+\frac{\beta}{2}\left(\rho-3p\right)^2}{1-\beta\left(\rho-3p\right)} \\ B=\dfrac{1}{6f_0}\dfrac{\rho-\frac{\beta}{4}\left(\rho-3p\right)^2}{1-\beta\left(\rho-3p\right)} \end{cases} \tag{44}$$

where $\beta=-\dfrac{f}{3{f_0}^2}$. System of equation (43) and solution (44) was found earlier in ref. [4], where the function Q_3 was not taken into consideration.

As a consequence of eq. (39) the torsion S is determined and equals to zero, if $a\neq 0$. It is easy to verify that solution (44) takes also place, if terms with coefficients m_i $(i=1,2,3)$ in $\mathcal{L}_G$ are taken into account and the following two conditions are imposed

$$m_1+3m_2=0,\quad m_1-3m_3=0. \tag{45}$$

Note, that solution (44) takes place in WCGT by the same restrictions on indefinite parameters of L_G.

From geometric point of view, obtained solution (44), (42) corresponds to the Weyl space-time and the nonmetricity is determined by the only function Q_1, which is the time component of the Weyl's vector. The solution (44) leads to the generalized cosmological Friedmann equation (GCFE) for the scale factor

$$\frac{k}{R^2}+\left\{\frac{d}{dt}\ln\left[R\sqrt{|1-\beta(\rho-3p)|}\,\right]\right\}^2=\frac{1}{6f_0}\frac{\rho-\frac{\beta}{4}\left(\rho-3p\right)^2}{1-\beta\left(\rho-3p\right)}. \tag{46}$$

GCFE (46) can be derived by using expressions (11) and (44) for the function B. At first, GCFE (46) was obtained in the frame of PGT [3]. Its validity in MAGT based on the gravitational Lagrangian with $k_i=0$ and $m_s=0$ was proved in [4]. It is important, that GCFE is a first order differential equation like the Friedmann cosmological equation of GR. GCFE satisfies the correspondence principle with GR in the case of sufficiently small energy densities $\rho\ll|\beta|^{-1}$, under which non-einsteinian characteristics (torsion and nonmetricity) are negligible.

Now let us investigate the system of gravitational equations of MAGT (28)–(33) for homogeneous isotropic models in the Weyl-Cartan space-time by using general gravitational Lagrangian (22)–(23). Then nonmetricity functions satisfy relations (42), and conditions (41) are assumed to be fulfilled. Equations (28) – (33) take the form

$$6f_0B-12f\left(A^2-B^2\right)+3aS^2-\left(k-\tfrac{1}{4}m\right)Q_1^2-\tfrac{3}{2}a\left(Q_1+2H\right)S+\tfrac{3}{2}mQ_1H=\rho, \tag{47}$$

$$\begin{aligned}&2f_0\left(2A+B\right)+4f\left(A^2-B^2\right)+aS^2+\left(k-\tfrac{1}{4}m\right)Q_1^2-2\left(m-\tfrac{1}{4}a\right)Q_1S+\\&+\left(mQ_1-2aS\right)H+\tfrac{1}{2}\left(m\dot{Q}_1-2a\dot{S}\right)=-p,\end{aligned} \tag{48}$$

$$\left(2k_1+2k_2+8k_3+2k_4+5k_5\right)Q_1-3\left(m_2+m_3\right)S=0, \tag{49}$$

$$2\left(k_2+k_4+2k_5+\frac{1}{8}m\right)Q_1+\left(m_1-3m_3-\frac{1}{2}a\right)S=0, \tag{50}$$

$$\left(4k-\frac{3}{2}m\right)Q_1-3\left(2m-a\right)S=0, \tag{51}$$

$$f\left(\dot{A}+\dot{B}\right)+\left[f_0+4f\left(A+B\right)\right]S_q+\frac{1}{4}\left(2aS-mQ_1\right)=0, \tag{52}$$

where the following notations are introduced

$$\begin{aligned} k &= 4k_1+k_2+16k_3+k_4+4k_5, \\ m &= m_1+4m_2+m_3. \end{aligned}$$

Note that only two equations from three (49)–(51) are linearly independent, therefore eq. (49) will be excluded from further consideration. The analysis of gravitational equations (47), (48), (50)–(52) leads us to the solution (44) corresponding to different models in the Riemann-Cartan, Weyl, Weyl-Cartan space-times in dependence on restrictions on parameters of L_G.

1. If $a=0$, $m=0$, $m_1-3m_3=0$, and $k\neq 0$ and/or $k_2+k_4+2k_5\neq 0$, we have $Q_1=0$ that corresponds to models in the Riemann-Cartan space-time considered at the first time in the frame of PGT [3].

2. If $k=m=k_2+k_4+2k_5=0$, and $a\neq 0$ and/or $m_1-3m_3-\frac{a}{2}\neq 0$, we have $S=0$ that corresponds to models in the Weyl space-time.

3. Isotropic models in the Weyl-Cartan space-time ($S\neq 0$, $Q_1\neq 0$) take place, if $a\neq 0$, $m\neq 0$ and $3m^2=4ak$ and, in addition, other restrictions on indefinite parameters are fulfilled:

   (a) $k_2+k_4+2k_5+\frac{m}{8}=0$, $m_1-3m_3-\frac{a}{2}=0$, $2m-a\neq 0$ $8k-3m\neq 0$ or

   (b) $3m(m_2+m_3)=2a(2k_1+2k_2+8k_3+2k_4+5k_5)$, $8k-3m\neq 0$, $2m-a\neq 0$, $k_2+k_4+2k_5+\frac{m}{8}\neq 0$, $m_1-3m_3-\frac{a}{2}\neq 0$.

   According to (44) and (51)

$$\left\{\begin{aligned} Q_1 &= \frac{a}{a-2m}\,\frac{d}{dt}\ln\left|1-\beta\left(\rho-3p\right)\right|, \\ S &= \frac{m}{2(a-2m)}\,\frac{d}{dt}\ln\left|1-\beta\left(\rho-3p\right)\right|. \end{aligned}\right. \tag{53}$$

Besides indicated two possibilities, solution for isotropic models in the Weyl-Cartan space-time can be obtained, if $k=a=m=0$, $k_2+k_4+2k_5\neq 0$, $m_1-3m_3\neq 0$, $m_1-3m_3+8(k_2+k_4+2k_5)\neq 0$, then

$$Q_1=-\frac{m_1-3m_3}{2(k_2+k_4+2k_5)}S$$

and according to (44)

$$\left\{\begin{aligned} Q_1 &= \frac{m_1-3m_3}{8(k_2+k_4+2k_5)+m_1-3m_3}\,\frac{d}{dt}\ln\left|1-\beta\left(\rho-3p\right)\right|, \\ S &= -\frac{2(k_2+k_4+2k_5)}{8(k_2+k_4+2k_5)+m_1-3m_3}\,\frac{d}{dt}\ln\left|1-\beta\left(\rho-3p\right)\right|. \end{aligned}\right.$$

In the frame of WCGT the system of gravitational equations for homogeneous isotropic models is reduced to (47), (48), (51) and (52). The solution (44) corresponds to different models in dependence on restrictions on parameters of L_G:

1. If $m = a = 0$ and $k \neq 0$ we have models in the Riemann-Cartan space-time with $S \neq 0$ and $Q_1 = 0$.

2. If $m = k = 0$ and $a \neq 0$ we have $S = 0$ and $Q_1 \neq 0$ that corresponds to models in in the Weyl space-time.

3. If $3m^2 - 4ak = 0$ and $k - \frac{3}{8}m \neq 0$, $2m - a \neq 0$ we have the models in the Weyl-Cartan space-time and the torsion and nonmetricity are defined by (53).

Note, that the metrics of all models indicated above satisfies the GCFE (46). This means that various regular cosmological solutions of GCFE obtained earlier in the frame of PGT (see refs. in [2]) take place also in the frame of MAGT (WCGT). Hence, the most important physical consequences obtained in the frame of PGT — the possible existence of limiting energy density for usual gravitating systems, the repulsion gravitational effect provoked by various physical factors etc. — are valid also in MAGT (WCGT).

References

[1] F.W.Hehl, G. D.McCrea, E.W.Mielke, and Y.Ne'eman, *Phys. Rep.*, **258**, 1 (1995).

[2] A.V.Minkevich, *Acta Phys. Polon.,* **29B**., No 4, 1, (1998).

[3] A.V.Minkevich, *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat.* **2**, 87; *Phys. Lett.* **80 A**, 232, (1980).

[4] A.V.Minkevich, *Dokl. Akad. Nauk Belarus*, **37**, 33, (1993).

[5] A.V.Minkevich, *Vestnik BGU, ser. 1*, No 3, 30, (1995).